Journal of Engineering Mathematics **32:** 73–86, 1997.

Concentrated force acting on an elastic inclusion in a thick-walled tube

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Received 4 July 1995; accepted in revised form 30 November 1996

Key words: Dual series expansion, indentation problem, elasticity, cylindrical tube

Abstract. A two-dimensional problem is investigated on the action of a concentrated force applied to the axis of a circular cylindrical, elastic inclusion embedded in an elastic thick-walled tube. This is a generalization of an indentation problem in infinite space, previously studied by Noble and Hussain [4] and revised by Omar and Hassan [5]. The problem is solved using a fast numerical approximation technique and numerical results are presented that allow us to evaluate the angle of contact and to establish a comparison with the case of embedding in an infinite space.

1. Introduction

Among the static problems of the Theory of Elasticity that have numerous applications in Engineering, the indentation problem is one of the most interesting. Due to its complexity, this problem can be solved exactly only in very few special cases and numerically in general cases [1]. The main difficulty in dealing with such problems arises from the fact that the contact region is not known *a priori*. For cylindrical regions of simple geometry, this amounts to saying that the contact angle is among the unknowns of the problem. Special methods were put forward to deal with, and to solve indentation problems ([2], and the literature cited therein), [3].

In their paper [4], Noble and Hussain reduce the problem of the inclusion of an infinite circular cylinder in an infinite space to that of solving an airfoil integral equation, under the constraint that the elastic parameters of the media satisfy a certain relation. The same problem was treated by Omar and Hassan [5] who used a simpler technique to solve the dual series equations to which the problem was reduced in the general case. They showed, in particular, that sufficiently accurate results may be obtained from the first few iterations of their solution without need to transform to the integral equation.

The problem is solved following the same technique as used in]5] and numerical results are given and discussed for the angle of contact between the inclusion and the tube. Comparison is established with the case of embedding in an infinite space. In particular, it is shown that the present results are the same as the corresponding ones in [5] when the shear modulus of the inclusion is much smaller than that of the outer medium.

2. Formulation of the problem

An infinite, isotropic, elastic circular thick-walled tube of radii a, $b(a < b)$ has an inclusion in the form of an infinite circular cylinder of radius a of another isotropic elastic material (see Fig. 1). A concentrated force F per unit length acts on the axis of the cylinder and

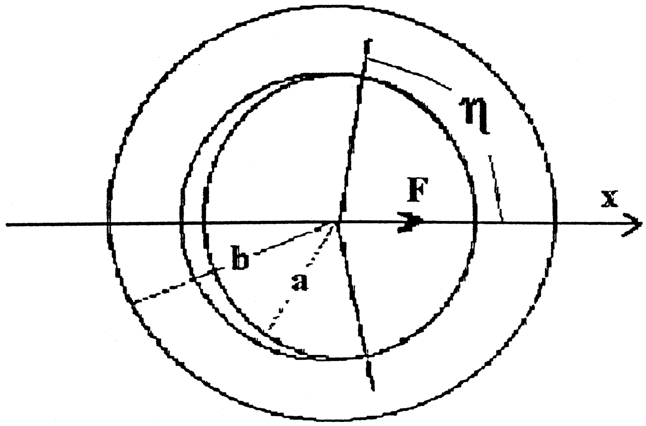

Figure 1. Geometry of the problem.

perpendicular to it. Accordingly, a separation region is established in the stressed medium, the end points of which need to be determined. It is well known that this problem reduces to the solution of a biharmonic equation for the stress function under proper conditions. This is further reduced to the solution of a pair of dual series equations involving the unknown angle of separation, the solution of which may be carried out numerically by means of an expansion in a small parameter, $\epsilon = a/b$, representing the ratio between the inner and outer radii of the tube. This permits the study of the case where the outer radius tends to infinity.

Let us introduce a set of cylindrical coordinates (r, θ, z) with the z-axis coinciding with the axis of the inclusion, the force acting along the polar axis $\theta = 0$. In what follows, we briefly quote the fundamental equations to be used in the sequel. The same notations as in [4] and [5] will be used.

(i) The stress components are expressed in terms of the stress function Φ as follows

$$\sigma_r = \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}, \quad \sigma_\theta = \frac{\partial^2\Phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right). \tag{1}$$

(ii) Strain-stress relations:

$$2G\epsilon_{rr} = (1-\nu)\sigma_r - \nu\sigma_\theta, \quad 2G\epsilon_{\theta\theta} = (1-\nu)\sigma_\theta - \nu\sigma_r, \quad 2G\epsilon_{r\theta} = \tau_{r\theta}, \tag{2}$$

where G and ν are the coefficient of rigidity and Poisson's ratio, respectively.

(iii) Strain-displacement relations:

$$\epsilon_{rr} = \frac{\partial u_r}{\partial_r}, \quad \epsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right), \quad 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}. \tag{3}$$

(iv) Boundary conditions:

If we assume a frictionless contact between the two bodies and a rigidly clamped outer surface of the tube, then the boundary conditions are as follows

$$\sigma_r(a,\theta) \;=\; \sigma'_r(a,\theta), \quad 0 \le \theta \le \pi, \tag{4}$$

$$\begin{aligned}
\tau_{r\theta}(a,\theta) &= \tau'_{r\theta}(a,\theta) = 0, \quad 0 \le \theta \le \pi, &(5)\\
u_r(a,\theta) &= u'_r(a,\theta), \qquad 0 \le \theta \le \eta, &(6)\\
\sigma_r(a,\theta) &= 0, \qquad \eta \le \theta \le \pi, &(7)\\
u_r(b,\theta) &= 0, \qquad 0 \le \theta \le \pi, &(8)\\
u_\theta(b,\theta) &= 0, \qquad 0 \le \theta \le \pi, &(9)
\end{aligned}$$

where the region of contact is $-\eta \le \theta \le \eta$, and the quantities referring to the inclusion are denoted by a ‘dash’, while the undashed quantities are for the tube.
The boundary conditions must be completed with the condition of univaluedness of the displacement

$$u_\theta(r,0) = u_\theta(r,\pi) = 0, \quad 0 \le r \le b. \tag{10}$$

Also, the following global equilibrium condition should hold for both the inclusion and the tube:

$$F = -2\int_0^\pi (\sigma_r \cos\theta - \tau_{r\theta}\sin\theta) r \, \mathrm{d}\theta. \tag{11}$$

From symmetry considerations, the stress functions Φ, Φ' may be shown to have the following expansions:

$$\begin{aligned}
\Phi' &= \frac{aF}{4\pi(1-\nu')}[(1-2\nu')\rho\log\rho\cos\theta - 2(1-\nu')\rho\theta\sin\theta]\\
&\quad + A'_0\rho^2 + A'_1\rho^3\cos\theta + \sum_{n=2}^{\infty}[A'_n\rho^{n+2} + B'_n\rho^n]\cos n\theta, &(12)
\end{aligned}$$

$$\begin{aligned}
\Phi &= \frac{aF}{4\pi(1-\nu)}[(1-2\nu)\rho\log\rho\cos\theta - 2(1-\nu)\rho\theta\sin\theta]\\
&\quad + A_0\rho^2 + [A_1\rho^3 + D_1\rho^{-1}]\cos\theta\\
&\quad + B_0\log\rho + a^2\sum_{n=2}^{\infty}[A_n\rho^{n+2} + B_n\rho^n + C_n\rho^{-n+2} + D_n\rho^{-n}]\cos n\theta, &(13)
\end{aligned}$$

where $\rho = r/a$ and $\{A'_n\}, \{B'_n\}, \{A_n\}, \{B_n\}, \{C_n\}, \{D_n\}$ are coefficients to be determined.
The following expressions for the stresses and displacements in the two media, satisfying (1)–(5) and (8)–(11), are finally obtained:
(i) For the inclusion $(0 \le \rho \le 1)$

$$\begin{aligned}
\sigma'_r &= \frac{1}{2}E_0 + \frac{E_1\cos\theta}{4(1-\nu')}[(1-2\nu')\rho + (3-2\nu')\rho^{-1}]\\
&\quad - \frac{1}{2}\sum_{n=2}^{\infty}[(n-2)\rho^n - n\rho^{n-2}]E_n\cos n\theta, &(14)
\end{aligned}$$

$$\sigma'_\theta = \frac{1}{2}E_0 + \frac{(1-2\nu')E_1\cos\theta}{4(1-\nu')}[3\rho - \rho^{-1}] + \frac{1}{2}\sum_{n=2}^{\infty}[(n+2)\rho^n - n\rho^{n-2}]E_n\cos n\theta, \tag{15}$$

$$\tau'_{r\theta} = \frac{(1-2\nu')E_1 \sin\theta}{4(1-\nu')}[\rho - \rho^{-1}] + \frac{1}{2}\sum_{n=2}^{\infty} n[\rho^n - \rho^{n-2}]E_n \sin n\theta, \tag{16}$$

$$\begin{aligned} 2G'\frac{u'_r}{a} \; = \; & \frac{1}{2}(1-2\nu')E_0\rho + \frac{2G'\delta}{a}\cos\theta + \frac{E_1\cos\theta}{8(1-\nu')}[(1-2\nu')(1-4\nu')\rho^2 + 2(3-4\nu')\log\rho] \\ & -\frac{1}{2}\sum_{n=2}^{\infty}\left[\frac{n-2+4\nu'}{n+1}\rho^{n+1} - \frac{n}{n-1}\rho^{n-1}\right] E_n \cos n\theta, \end{aligned} \tag{17}$$

$$\begin{aligned} 2G'\frac{u'_\theta}{a} \; = \; & -\frac{2G'\delta}{a}\sin\theta + \frac{E_1\sin\theta}{8(1-\nu')}[(1-2\nu')(5-4\nu')\rho^2 - 2 - 2(3-4\nu')\log\rho] \\ & +\frac{1}{2}\sum_{n=2}^{\infty}\left[\frac{n+4-4\nu'}{n+1}\rho^{n+1} - \frac{n}{n-1}\rho^{n-1}\right] E_n \sin n\theta. \end{aligned} \tag{18}$$

(ii) For the tube $(1 \leq \rho \leq 1/\epsilon)$

$$\begin{aligned} \sigma_r \; = \; & \frac{1}{2}E_0\frac{\epsilon^2 + (1-2\nu)\rho^{-2}}{(1-2\nu)+\epsilon^2} \\ & +\frac{E_1\cos\theta}{4(1-\nu)}\left[\frac{(1-2\nu)(3-4\nu)-\epsilon^2}{(3-4\nu)+\epsilon^4}\rho^{-3} + (3-2\nu)\rho^{-1} + \frac{(1-2\nu)\epsilon^4+\epsilon^2}{(3-4\nu)+\epsilon^4}\rho\right] \\ & -\sum_{n=2}^{\infty}[(n+1)(n-2)A_n\rho^n + n(n-1)B_n\rho^{n-2} \\ & +(n-1)(n+2)C_n\rho^{-n} + n(n+1)D_n\rho^{-n-2}]\cos n\theta, \end{aligned} \tag{19}$$

$$\begin{aligned} \sigma_\theta \; = \; & \frac{1}{2}E_0\frac{\epsilon^2 - (1-2\nu)\rho^{-2}}{(1-2\nu)+\epsilon^2} \\ & -\frac{E_1\cos\theta}{4(1-\nu)}\left[\frac{(1-2\nu)(3-4\nu)-\epsilon^2}{(3-4\nu)+\epsilon^4}\rho^{-3} + (1-2\nu)\rho^{-1} - 3\frac{(1-2\nu)\epsilon^4+\epsilon^2}{(3-4\nu)+\epsilon^4}\rho\right] \\ & +\sum_{n=2}^{\infty}[(n+1)(n+2)A_n\rho^n + n(n-1)B_n\rho^{n-2} \\ & +(n-1)(n-2)C_n\rho^{-n} + n(n+1)D_n\rho^{-n-2}]\cos n\theta, \end{aligned} \tag{20}$$

$$\begin{aligned} \tau_{r\theta} \; = \; & \frac{E_1\sin\theta}{4(1-\nu)}\left[\frac{(1-2\nu)(3-4\nu)-\epsilon^2}{(3-4\nu)+\epsilon^4}\rho^{-3} - (1-2\nu)\rho^{-1} + \frac{(1-2\nu)\epsilon^4+\epsilon^2}{(3-4\nu)+\epsilon^4}\rho\right] \\ & +\sum_{n=2}^{\infty} n[(n+1)A_n\rho^n + (n-1)B_n\rho^{n-2} - (n-1)C_n\rho^{-n} \\ & -(n+1)D_n\rho^{-n-2}]\sin n\theta, \end{aligned} \tag{21}$$

$$2G\frac{u_r}{a} \; = \; \frac{1}{2}\frac{1-2\nu}{1-2\nu+\epsilon^2}[\epsilon^2\rho - \rho^{-1}]E_0$$

$$+\frac{E_1\cos\theta}{8(1-\nu)}\left[2(3-4\nu)\log\epsilon\rho+\frac{\Gamma_1\rho^{-2}+(1-4\nu)\Gamma_2\rho^2-\Gamma_3}{(3-4\nu)+\epsilon^4}\right]$$
$$-\sum_{n=2}^{\infty}[(n-2+4\nu)A_n\rho^{n+1}+nB_n\rho^{n-1}$$
$$-(n+2-4\nu)C_n\rho^{-n+1}-nD_n\rho^{-n-1}]\cos n\theta, \tag{22}$$

$$2G\frac{u_\theta}{a} = \frac{E_1\sin\theta}{8(1-\nu)}\left[-2(3-4\nu)\log\epsilon\rho-2+\frac{\Gamma_1\rho^{-2}+(5-4\nu)\Gamma_2\rho^2+\Gamma_3}{(3-4\nu)+\epsilon^4}\right]$$
$$+\sum_{n=2}^{\infty}[(n+4-4\nu)A_n\rho^{n+1}+nB_n\rho^{n-1}+(n-4+4\nu)C_n\rho^{-n+1}$$
$$+nD_n\rho^{-n-1}]\sin n\theta, \tag{23}$$

where δ is the rigid-body displacement of the inclusion in the force direction and the coefficients are interrelated by the relations:

$$\Gamma_1=[\epsilon^2-(1-2\nu)(3-4\nu)], \quad \Gamma_2=\epsilon^2[(1-2\nu)\epsilon^2+1],$$

$$\Gamma_3=(1-4\nu)-2(1-2\nu)\epsilon^2+\epsilon^4,$$

$$E_1=-\frac{F}{\pi a}, \qquad A'_0=\frac{1}{4}a^2E_0, \qquad A'_1=\frac{(1-2\nu')a^2}{8(1-\nu')}E_1,$$

$$A_0=\frac{a^2\epsilon^2E_0}{4[(1-2\nu)+\epsilon^2]}, \qquad A_1=\frac{a^2E_1}{8(1-\nu)}\frac{(1-2\nu)\epsilon^4+\epsilon^2}{(3-4\nu)+\epsilon^4},$$

$$B_0=\frac{1-2\nu}{1-2\nu+\epsilon^2}a^2\frac{E_0}{2}, \qquad D_1=-\frac{a^2E_1}{8(1-\nu)}\frac{(1-2\nu)(3-4\nu)-\epsilon^2}{(3-4\nu)+\epsilon^4}$$

and for $(n\geq 2)$:

$$A'_n=\frac{a^2E_n}{2(n+1)}, \qquad B'_n=-\frac{a^2E_n}{2(n-1)},$$

$$(n+1)(n-2)A_n+n(n-1)B_n+(n-1)(n+2)C_n+n(n+1)D_n=-E_n,$$
$$n(n+1)A_n+n(n-1)B_n-n(n-1)C_n-n(n+1)D_n=0,$$
$$(n-2+4\nu)\epsilon^{-n-1}A_n+n\epsilon^{1-n}B_n-(n+2-4\nu)\epsilon^{n-1}C_n-n\epsilon^{n+1}D_n=0,$$
$$(n+4-4\nu)\epsilon^{-n-1}A_n+n\epsilon^{1-n}B_n+(n-4+4\nu)\epsilon^{n-1}C_n+n\epsilon^{n+1}D_n=0.$$

The boundary conditions (6) and (7) give the following dual series equations in the unknowns $E_0, E_n (n\geq 2), \delta$ and η:

$$\frac{1}{2}E_0+\sum_{n=1}^{\infty}E_n\cos n\theta=0, \qquad \eta\leq\theta\leq\pi, \tag{24}$$

$$\frac{1}{2}k_0 E_0 + cE_1\cos\theta + \sum_{n=2}^{\infty}\frac{n-\kappa_n}{n^2-1}E_n\cos n\theta = -\alpha\delta'\cos\theta, \quad 0\le\theta\le\eta, \tag{25}$$

where

$$k_0 = \frac{K_0}{2K_2}, \quad \kappa_n = \frac{K_1 - S_n}{2K_2}, \quad c = \frac{K_3}{2K_2}, \quad \delta' = \frac{G`\delta}{aK_2}, \quad \alpha = \frac{G}{G'},$$

$$\begin{aligned}
K_0 &= \alpha(1-2\nu') + \frac{(1-2\nu)(1-\epsilon^2)}{(1-2\nu)+\epsilon^2},\\
K_1 &= \alpha(1-2\nu') - (1-2\nu),\\
K_2 &= \alpha(1-\nu') + (1-\nu),\\
K_3 &= \alpha L' + L,\\
L &= \frac{2\nu(3-4\nu)\epsilon^4 - 4(1-2\nu)\epsilon^2 + (1-2\nu)(3-4\nu) + 1 - 4\nu}{8(1-\nu)[(3-4\nu)+\epsilon^4]} - \frac{(3-4\nu)\log\epsilon}{4(1-\nu)},\\
L' &= \frac{(1-2\nu')(1-4\nu')}{8(1-\nu')},
\end{aligned}$$

$$\begin{aligned}
S_2 &= \frac{4(1-\nu)}{3-4\nu}\epsilon^2\Bigg\{6[1+2(1-\nu)(1-2\nu)] - 6\left(1+\frac{4[1+2(1-\nu)(1-2\nu)]^2}{3-4\nu}\right)\epsilon^2\\
&\quad +2\left(1+\frac{30[1+2(1-\nu)(1-2\nu)]}{3-4\nu} + \frac{48[1+2(1-\nu)(1-2\nu)]^3}{(3-4\nu)^2}\right)\epsilon^4\\
&\quad +2\Bigg((3-4\nu) - \frac{18+16[1+2(1-\nu)(1-2\nu)]}{3-4\nu} - \frac{192[1+2(1-\nu)(1-2\nu)]^2}{(3-4\nu)^2}\\
&\quad -\frac{192[1+2(1-\nu)(1-2\nu)]^4}{(3-4\nu)^3}\Bigg)\epsilon^6 + O(\epsilon^8)\Bigg\},\\
S_3 &= \frac{4(1-\nu)}{3-4\nu}\epsilon^4\Bigg\{2[9+8(1-\nu)(1-2\nu)] - 24\epsilon^2\\
&\quad +\left(9-\frac{2[9+8(1-\nu)(1-2\nu)]^2}{3-4\nu}\right)\epsilon^4 + O(\epsilon^6)\Bigg\},\\
S_4 &= \frac{4(1-\nu)}{3-4\nu}\epsilon^6\{20[2+(1-\nu)(1-2\nu)] - 60\epsilon^2 + O(\epsilon^4)\},\\
S_5 &= \frac{4(1-\nu)}{3-4\nu}\epsilon^8\{3[25+8(1-\nu)(1-2\nu)] + O(\epsilon^2)\},\\
S_n &= O(\epsilon^{2(n-1)}), \quad n\ge 6.
\end{aligned}$$

3. Solution of the dual series equations

To find the approximate solution of the dual series equations (24) and (25), we make use of the method suggested in [5]. Applying the operator $(D+D^{-1})$ on equation (25), where

$$Df = \frac{\mathrm{d}f}{\mathrm{d}\theta}, \quad D^{-1}f = \int_0^{\theta} f(\theta)\,\mathrm{d}\theta,$$

we get

$$-\frac{1}{2}k_0E_0\theta+\sum_{n=2}^{\infty}E_n\left(1-\frac{\kappa_n}{n}\right)\sin n\theta=0,\quad 0\le\theta\le\eta. \tag{26}$$

Equations (24) and (26) are sufficient to determine all the unknowns. Choosing any integer $M\ge 1$, we can rewrite Eq. (26) in the truncated form

$$-\frac{1}{2}k_0E_0\theta+\sum_{n=1}^{\infty}E_n\sin n\theta=E_1\sin\theta+\sum_{m=2}^{M+1}\frac{\kappa_m}{m}E_m\sin m\theta,\quad 0\le\theta\le\eta. \tag{27}$$

Thus, the M-th order approximate solution of the dual series equations (24) and (27) will be

$$E_n=a_{n1}E_1+\sum_{m=2}^{M+1}\kappa_m a_{nm}E_m,\quad n=0,1,2,\ldots, \tag{28}$$

where $\{a_{ij}\}$ is the solution of the pair of dual series equations

$$\begin{aligned}-\frac{1}{2}k_0a_{0m}\theta+\sum_{n=1}^{\infty}a_{nm}\sin n\theta&=\frac{\sin m\theta}{m},\quad 0\le\theta\le\eta,\\ \frac{1}{2}a_{0m}+\sum_{n=1}^{\infty}a_{nm}\cos n\theta&=0,\quad \eta\le\theta\le\pi,\end{aligned} \tag{29}$$

which is given in [5].

Equations (28), for n running over the set of values $1,2,\ldots,M+1$, form a set of $(M+1)$ homogeneous linear, algebraic equations in $E_n(1\le n\le M+1)$. Since $E_1\ne 0$, the determinant of the matrix of this system of equations must vanish, from which we can determine the angle η for the M-th order of approximation. We then calculate the values of the coefficients E_0 and $\{E_n\}(n\ge 2)$ as in [5]. The parameter δ' may be determined from Equation (25) with $\theta=0$.

4. Numerical results and discussion

Some numerical calculations for the angle of contact η were carried out. Each of the Figures 2–4 shows the curves of the angle η against the physical parameter $z=\alpha/(\alpha+1)$ for different values of the geometrical parameter ϵ and for definite values of ν and ν'.

For the sake of comparison with the results of [5], we have plotted in Figures 5–8 the difference $\Delta\eta$ between the actual angle and the corresponding one for the case $\epsilon=0$ (case treated in [5]).

The results show that:

1. When $G'\ll G$, the angle of contact is independent of ϵ.
2. As the parameter ϵ increases, i.e. as the tube becomes thinner, the angle of contact increases monotonically.
3. When $\nu'=0.5$, i.e. when the material of the inclusion is incompressible, the difference $\Delta\eta$ is almost constant, as long as $\alpha\le 4$.
4. For $\epsilon=a/b\le 0.3$, the difference $\Delta\eta$ in general does not exceed 10°, whatever the values of ν,ν' and α.
5. When $\alpha\to\infty$, i.e. when $G'\ll G$, $\Delta\eta\to 0$, whatever the values of ν,ν' and ϵ.

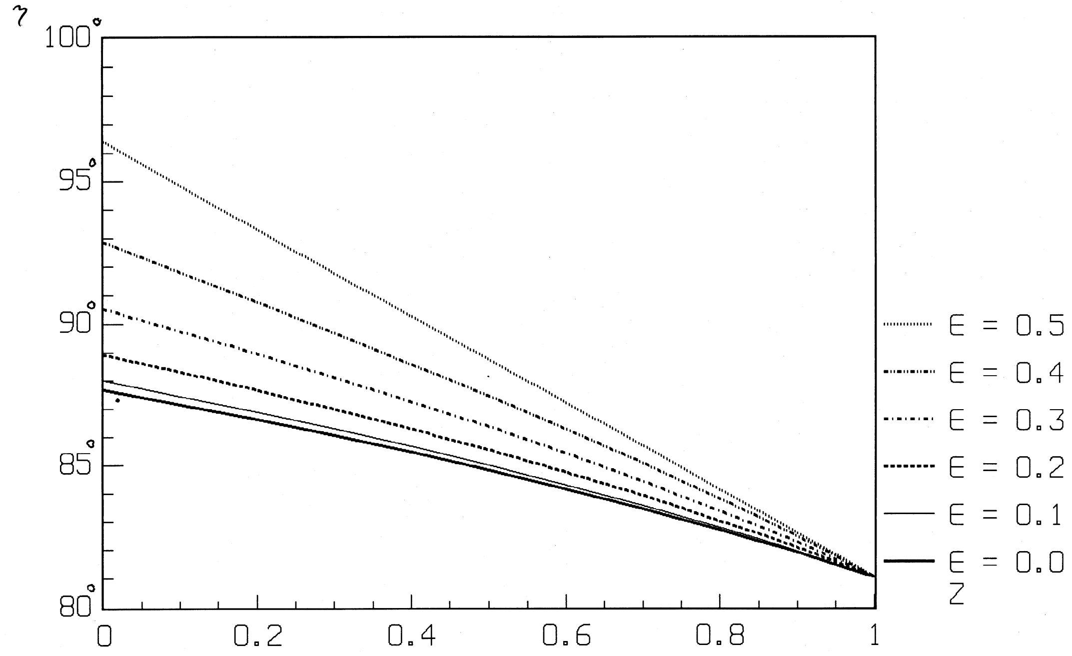

Figure 2. Values of η for $\nu = 0.0$, $\nu' = 0.0$.

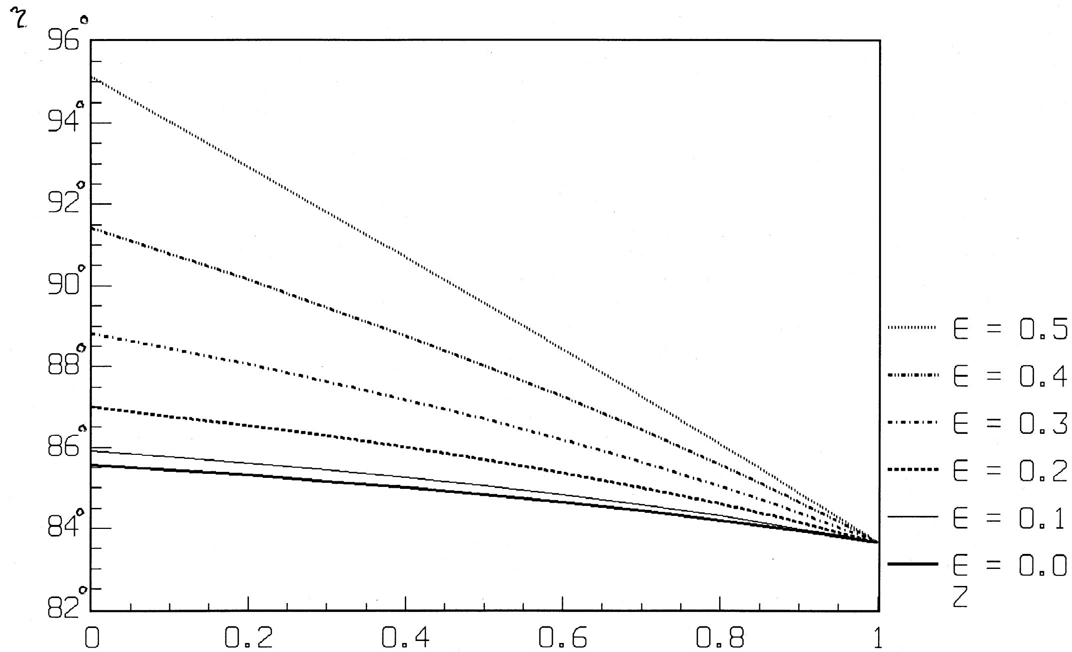

Figure 3. Values of η for $\nu = 0.1$, $\nu' = 0.1$.

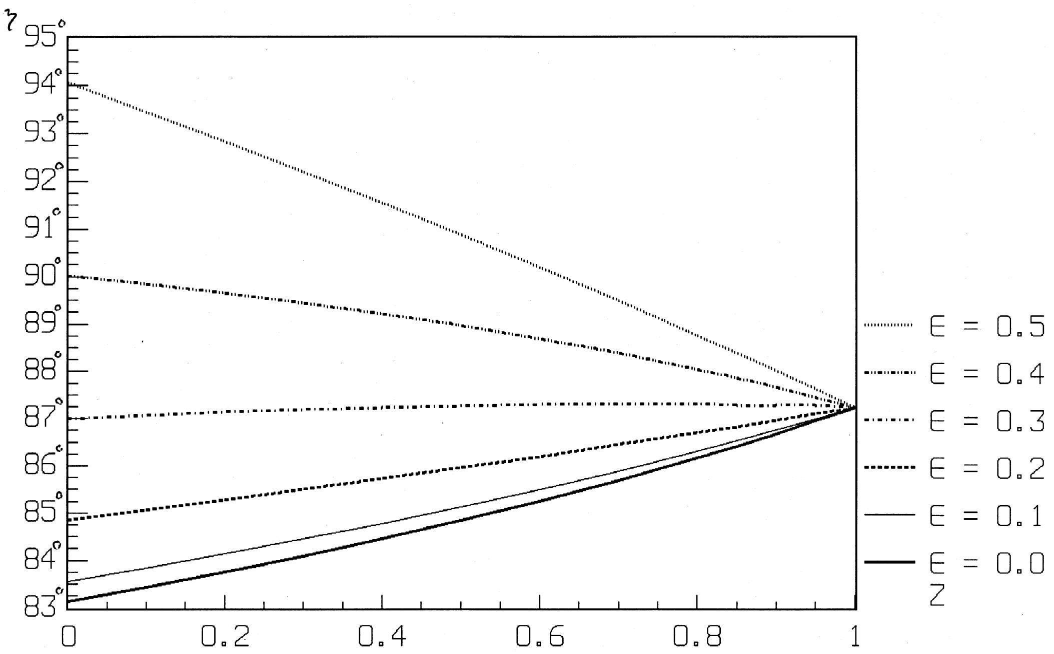

Figure 4. Values of η for $\nu = 0.2$, $\nu' = 0.2$.

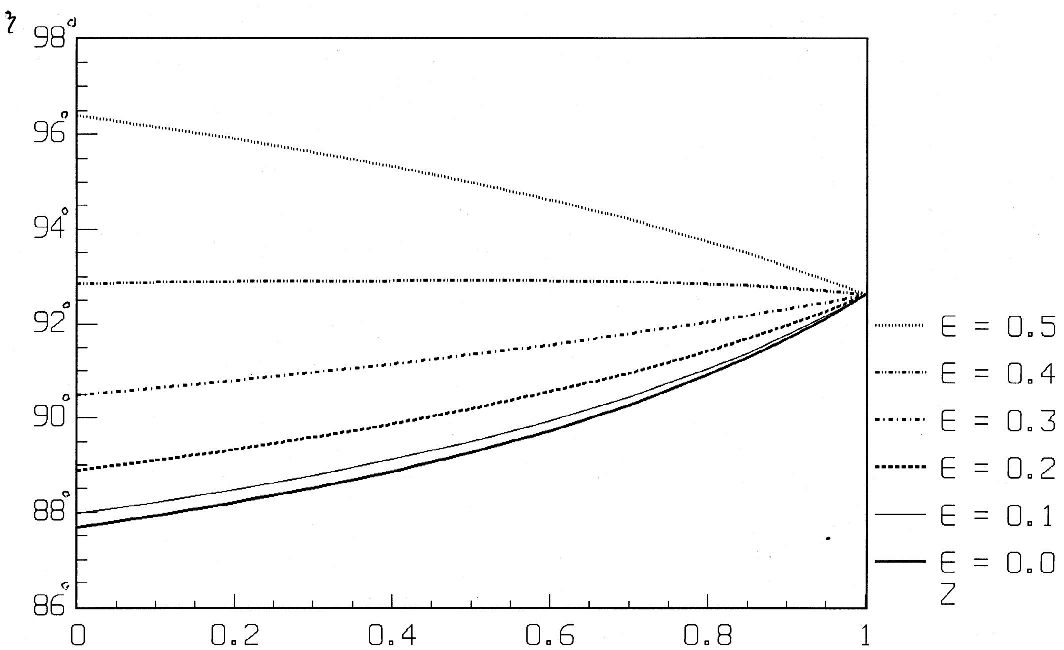

Figure 5. Values of η for $\nu = 0.0$, $\nu' = 0.3$.

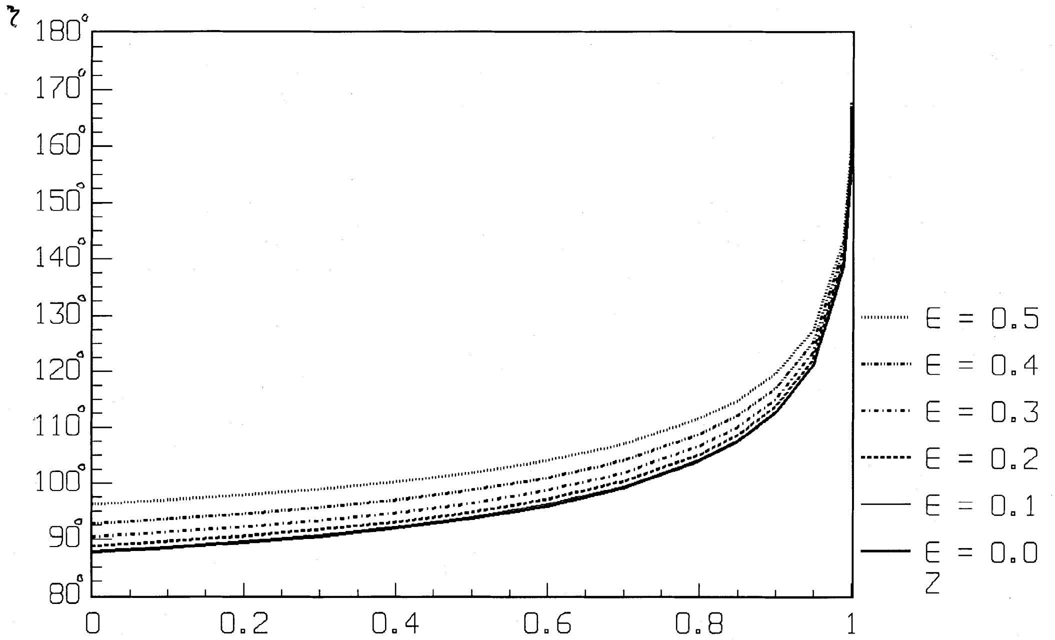

Figure 6. Values of η for $\nu = 0.0$, $\nu' = 0.5$.

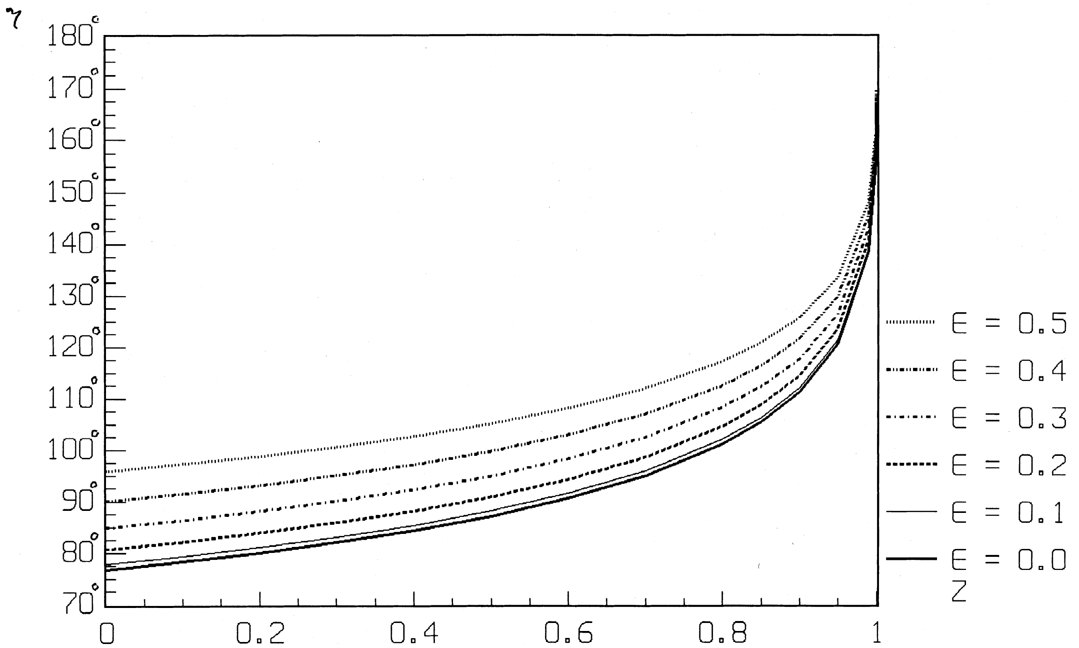

Figure 7. Values of η for $\nu = 0.4$, $\nu' = 0.5$.

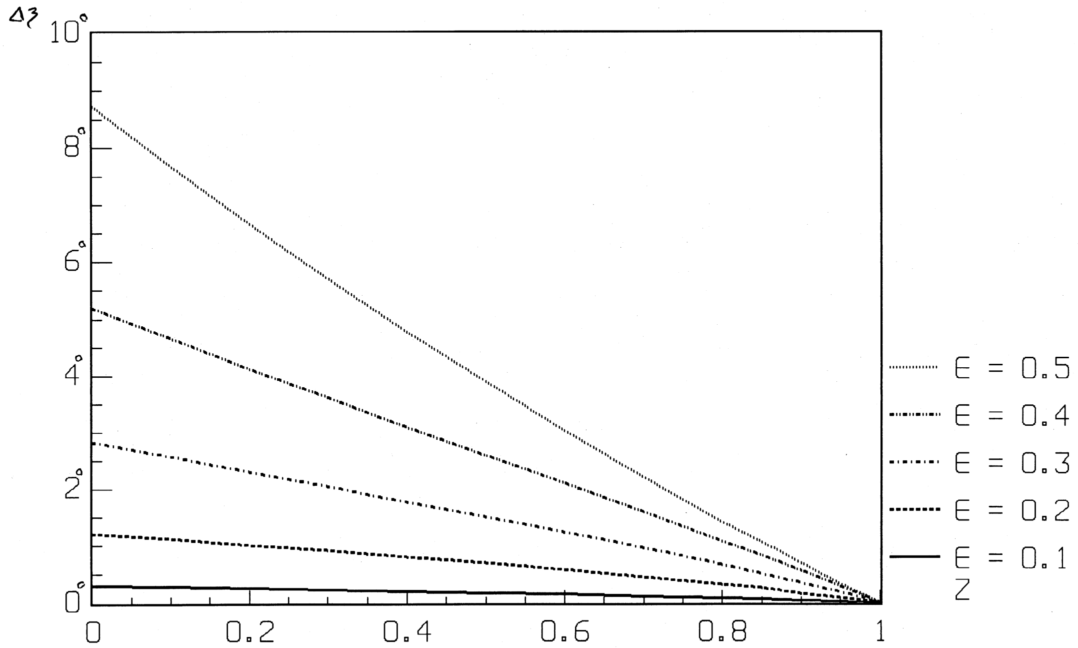

Figure 8. Values of $\Delta\eta$ for $\nu = 0.0$, $\nu' = 0.0$.

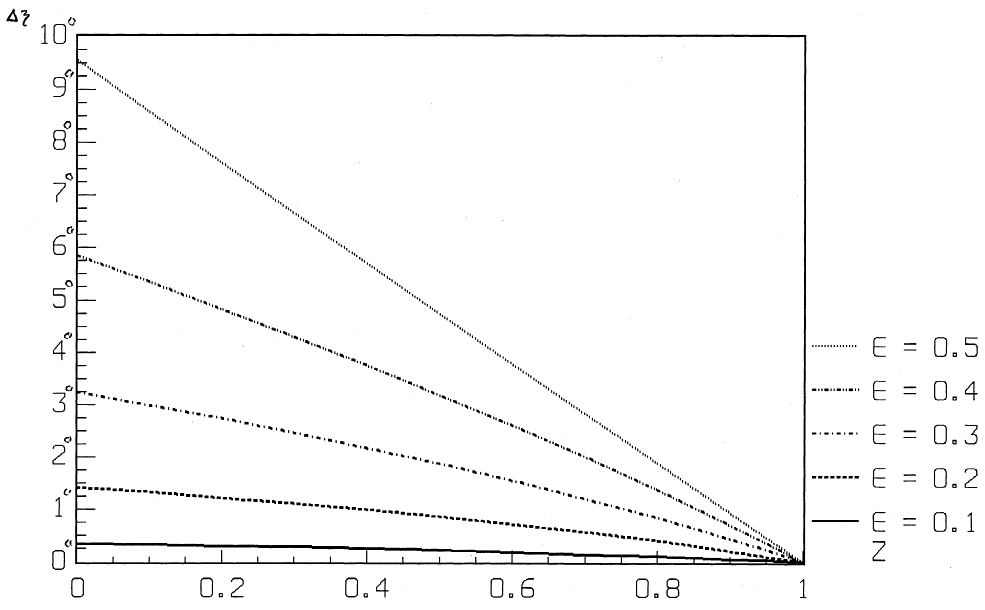

Figure 9. Values of $\Delta\eta$ for $\nu = 0.1$, $\nu' = 0.1$.

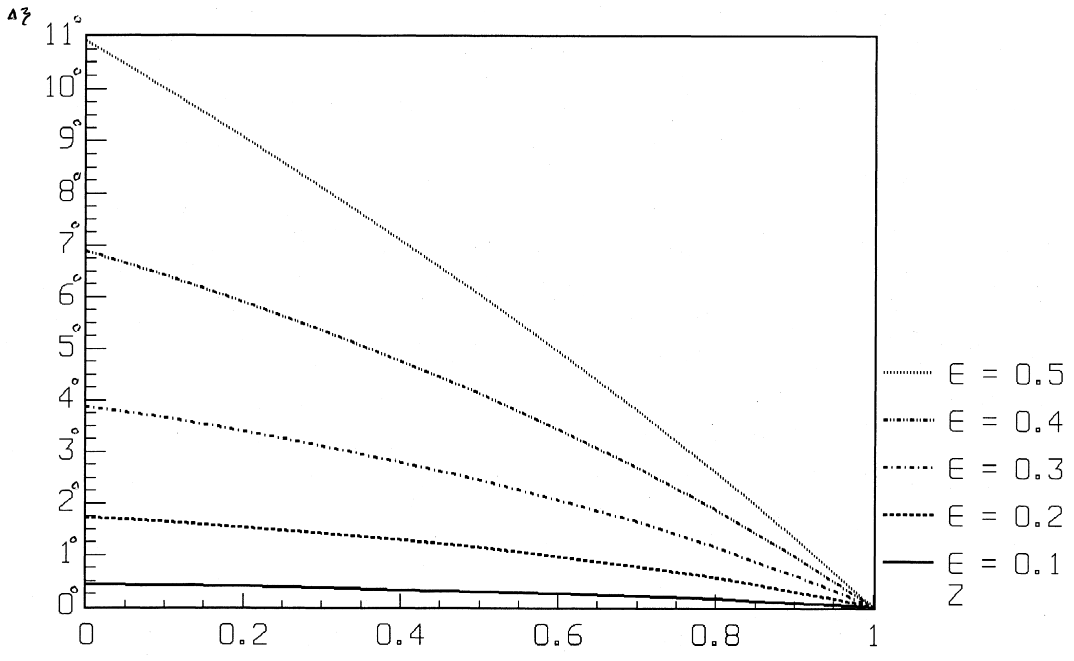

Figure 10. Values of $\Delta\eta$ for $\nu = 0.2$, $\nu' = 0.2$.

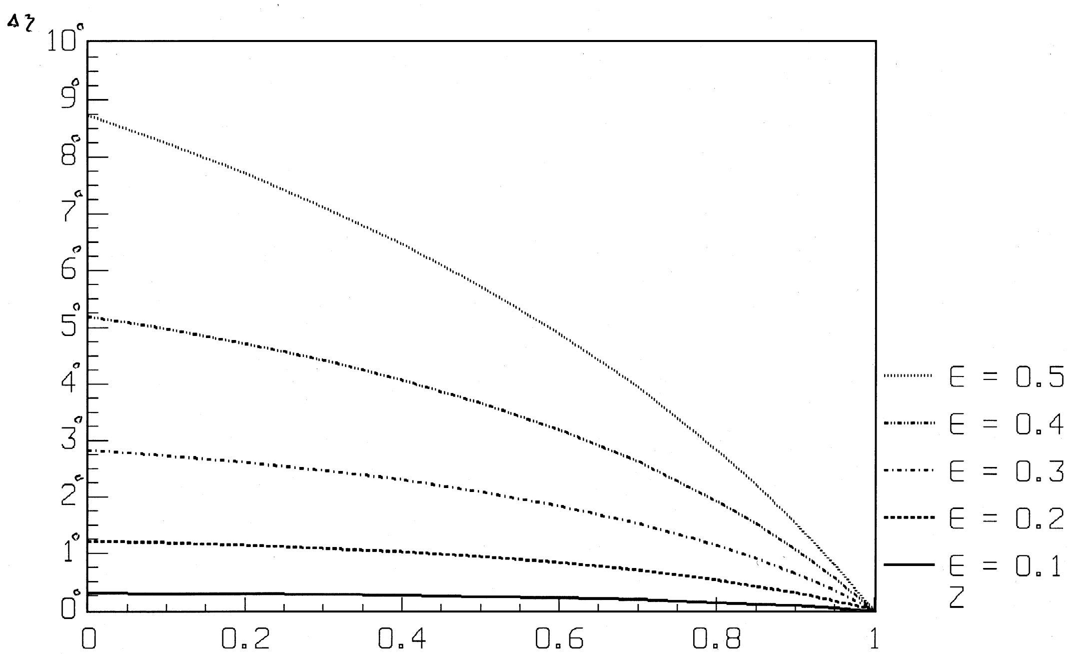

Figure 11. Values of $\Delta\eta$ for $\nu = 0.0$, $\nu' = 0.3$.

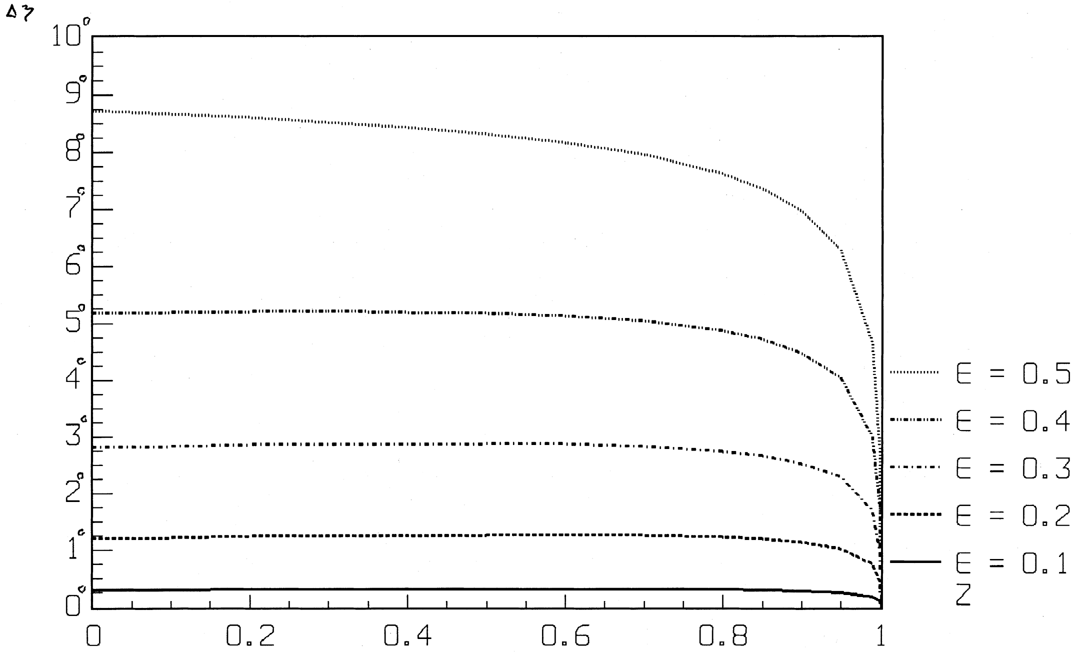

Figure 12. Values of $\Delta\eta$ for $\nu = 0.0$, $\nu' = 0.5$.

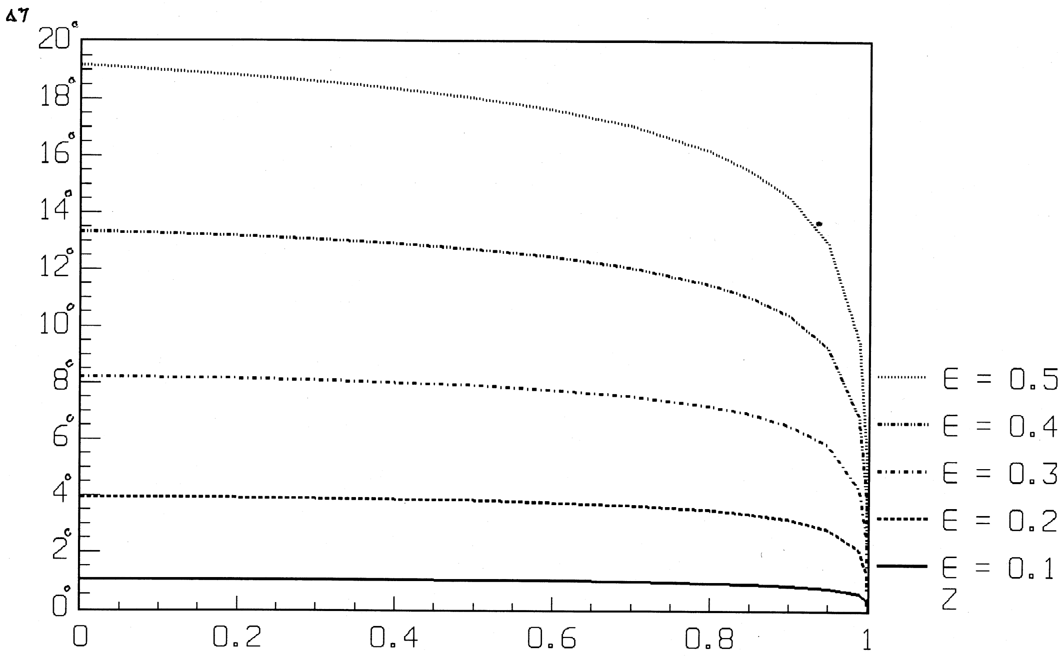

Figure 13. Values of $\Delta\eta$ for $\nu = 0.4$, $\nu' = 0.5$.

5. Acknowledgements

The referees are gratefully acknowledged for their careful reading and useful suggestions.

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